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Extension of the Arrowsmith–Essam Formula to the Domany–Kinzel Model

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Arrowsmith and Essam gave an expansion formula for point-to-point connectedness functions of the mixed site-bond percolation model on oriented lattices, in which each term is characterized by a graph. We extend this formula to general k-point correlation functions, which are point-to-set (with k points) connectivities in the context of percolation, of the two-neighbor discrete-time Markov process (stochastic cellular automata with two parameters) in one dimension called the Domany–Kinzel model, which includes the mixed site-bond oriented percolation model on a square lattice as a special case. Our proof of the formula is elementary and based on induction with respect to time-step, which is different from the original graph-theoretical one given by Arrowsmith and Essam. We introduce a system of m interacting random walkers called m friendly walkers (m FW) with two parameters. Following the argument of Cardy and Colaiori, it is shown that our formula is useful to derive a theorem that the correlation functions of the Domany–Kinzel model are obtained as an $m \to 0$ limit of the generating functions of the m FW.

KEY WORDS: Arrowsmith–Essam formula; Domany–Kinzel model; oriented percolation; friendly walkers.

1. INTRODUCTION AND MAIN RESULT

In this paper we consider the following two-neighbor discrete-time Markov process ξ_n^A with time n starting from $A \subset 2\mathbf{Z}$ on the collection of finite subsets of integers $\mathbf{Z}$ with the following evolution:

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(i) given ξ_n^A, the events $\{x \in \xi_{n+1}^A\}$ are independent, and

(ii) $P(x \in \xi_{n+1}^A \mid \xi_n^A) = f(|\xi_n^A \cap \{x-1, x+1\}|)$, where

$$f(0)=0, \qquad f(1)=p, \qquad \text{and} \qquad f(2)=q$$

with $p, q \in [0, 1]$, $P(B \mid C)$ is the conditional probability of an event B given an event C, and $|A|$ is the cardinality (the number of elements) of a set A. Figure 1 illustrates the local rule of the model. This process can be represented by a random configuration on a spatio-temporal plane $V = \{v=(x, n) \in \mathbf{Z} \times \mathbf{Z}_+ : x+n = \text{even}\}$, where $\mathbf{Z}_+ = \{0, 1, 2,...\}$. In V we use a norm $|v| = |x| + |n|$. This class of stochastic cellular automata was first studied by Domany and Kinzel,[1] and we call this the Domany–Kinzel model in the present paper.

The oriented site percolation ($q=p$) and oriented bond percolation ($q=2p-p^2$) are special cases. The mixed site-bond oriented percolation with the probabilities of an open site α and of an open bond β corresponds to the case of $p=\alpha\beta$ and $q=\alpha(2\beta-\beta^2)$. See Section 5 in Durrett[2] for details.

In the case of the mixed site-bond percolation with parameters α and β, Arrowsmith and Essam[3] proved a formula which is given as a special case with $k=1$ (i.e., for a pair connectedness function) of (1.5) below. In the present paper, we extend this *Arrowsmith–Essam formula* to the general case of the Domany–Kinzel model for any k-point correlation functions (Theorem 1). Our method to prove the theorem is elementary and based on induction with respect to time-step n. It is different from the original one given by Arrowsmith and Essam which is based mainly on graph theoretical argument.[3]

To state our results, we need to introduce some definitions and notations. For any $m \in \{0, 1,...\}$, we introduce a *level m* as $V_m = \{v=(x, n) \in V : n=m\}$. Then $V = \bigcup_{m=0}^{\infty} V_m$.

From now on, we consider only the case of the process ξ_n^0 starting from the origin $\{0\}$. A realization of $\xi_\cdot^0$ can be represented by a configuration

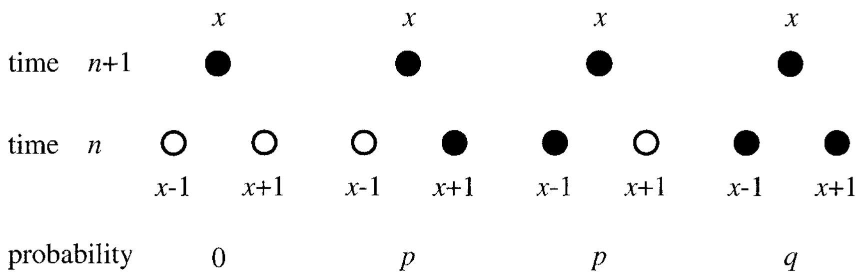

Fig. 1. The local stochastic rule of time-evolution of the Domany–Kinzel model. The full (resp. open) circle denotes the site included (resp. not included) in $\xi_\cdot^A$.

$\eta \in \{0, 1\}^{\mathbf{T}}$, where $\mathbf{T} = \{v = (x, n) \in V : |x| \leqslant n\} = \{(0, 0), (-1, 1), (1, 1), (-2, 2), (0, 2), (2, 2),...\}$. Therefore, $\xi_m^0 = \{x_1, x_2,..., x_k\}$ is equivalent to $\eta(x_1, m) = \eta(x_2, m) = \cdots = \eta(x_k, m) = 1$ and $\eta(y, m) = 0$ for $y \neq x_i\ (1 \leqslant i \leqslant k)$, where $\eta(x, n) = \eta((x, n))$.

For each $n \geqslant 1$, define

$$P_n(x_1, x_2,..., x_k) = P(\eta(x_1, n) = \eta(x_2, n) = \cdots = \eta(x_k, n) = 1) \tag{1.1}$$

where $x_1 < x_2 < \cdots < x_k$. We call this probability the *k-point correlation function* of the Domany–Kinzel model. For example, consider the case $n = 2$, $x_1 = -2$ and $x_2 = 0$ as shown in Fig. 2a. There are two disjoint events A and B, which are subsets of $\{0, 1\}^{\mathbf{T}}$, given in Fig. 2b, contributing to $P_2(-2, 0)$. Since $P(A) = p \times (1-p) \times p \times p = p^3(1-p)$ and $P(B) = p \times p \times p \times q = p^3 q$, we have

$$P_2(-2, 0) = P(A) + P(B) = (1+q)\, p^3 - p^4$$

For a pair of sites $(v_i, v_{i+1}) \in V_m \times V_{m+1}$, $m \geqslant 0$, we put a bond, which is an oriented arc from v_i to v_{i+1}, if and only if $|v_{i+1} - v_i| = 2$. A sequence of successive bonds $\{(v_0, v_1), (v_1, v_2),..., (v_{n-1}, v_n)\}$ with $v_m \in V_m$, $0 \leqslant m \leqslant n$, is called a *path* from v_0 to v_n and is denoted by $\pi(v_0, v_n)$. Let O be the origin and $\mathscr{A} = \{(x_1, n), (x_2, n),..., (x_k, n)\} \subset V_n$ with $x_1 < x_2 < \cdots < x_k$, where $n \geqslant 1$ and $n+1 \geqslant k \geqslant 1$ are assumed. Consider a collection of $r(\geqslant k)$ distinct paths $\{\pi_1(O, v_1), \pi_2(O, v_2),..., \pi_r(O, v_r)\}$ such that $\{v_1, v_2,..., v_r\} = \mathscr{A}$. We regard each path $\pi_i(O, v_i)$ as a set of bonds and take a union of all paths in this collection,

$$g = \bigcup_{i=1}^{r} \pi_i(O, v_i)$$

which is a graph on $\mathbf{T}$. We simply call such a graph, which is constructed by the above mentioned procedure, *a graph connecting O and $\mathscr{A}$ made of r paths* and define $\mathscr{G}_n^{(r)}(x_1, x_2,..., x_k)$ be the set of all graphs connecting O and $\{(x_1, n), (x_2, n),..., (x_k, n)\}$ made of r paths. Then we define

$$\mathscr{G}_n(x_1, x_2,..., x_k) = \bigcup_{r \geqslant k} \mathscr{G}_n^{(r)}(x_1, x_2,..., x_k)$$

whose elements are simply called *graphs connecting O and* $\{(x_1, n), (x_2, n),..., (x_k, n)\}$. For example, Fig. 3a shows three graphs, g_1, g_2 and g_3, included in $\mathscr{G}_2(-2, 0)$. We also show the graphs, g_4, g_5, g_6, in Fig. 3b which are *not* included in $\mathscr{G}_2(-2, 0)$. For any $g \in \mathscr{G}_n(x_1, x_2,..., x_k)$, let $\ell(g)$ and $b(g)$ denote the number of loops and bonds in g, respectively. The values of $\ell(g)$ and $b(g)$ are listed in Fig. 3a for $\mathscr{G}_2(-2, 0)$.

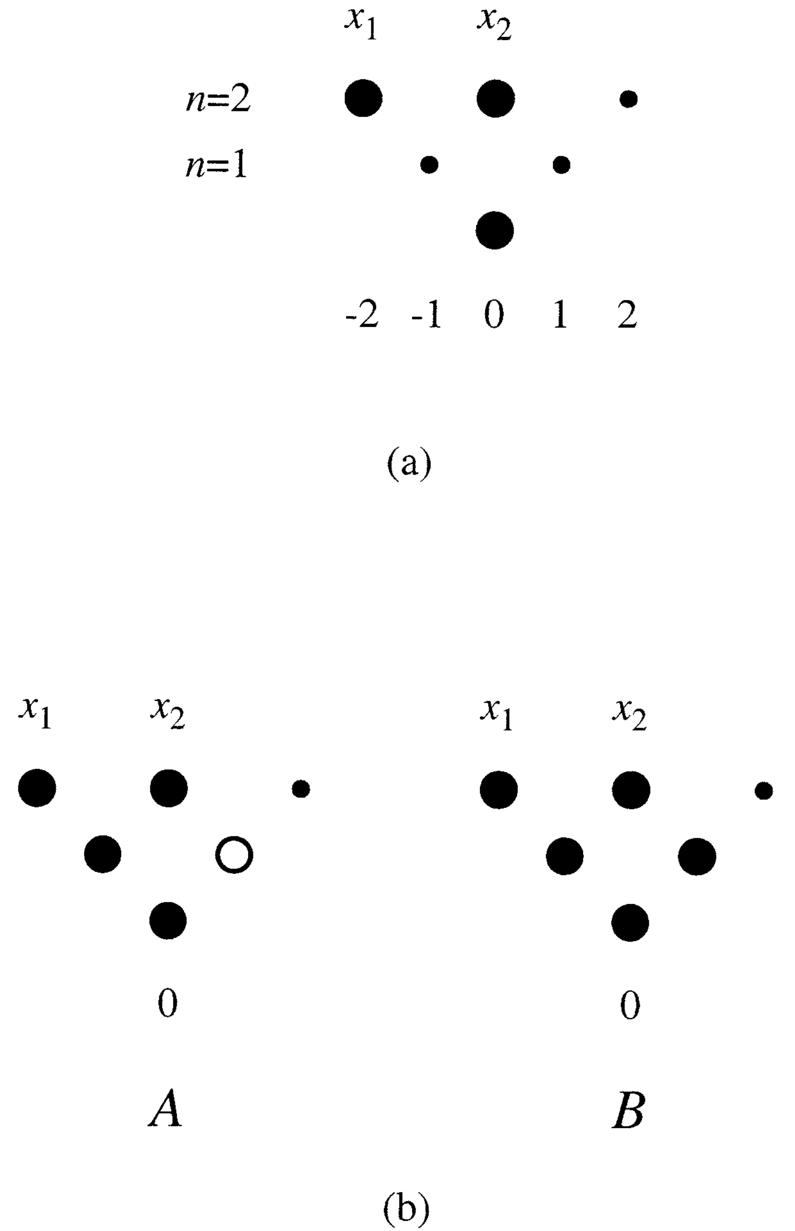

Fig. 2. (a) A figure for $P_2(-2, 0)$. (b) Two events A and B which contribute to $P_2(-2, 0)$. The full (resp. open) circle denotes the site (x, n) at which $\eta_i(x, n) = 1$ (resp. 0). Configurations on the sites shown by dots are not fixed.

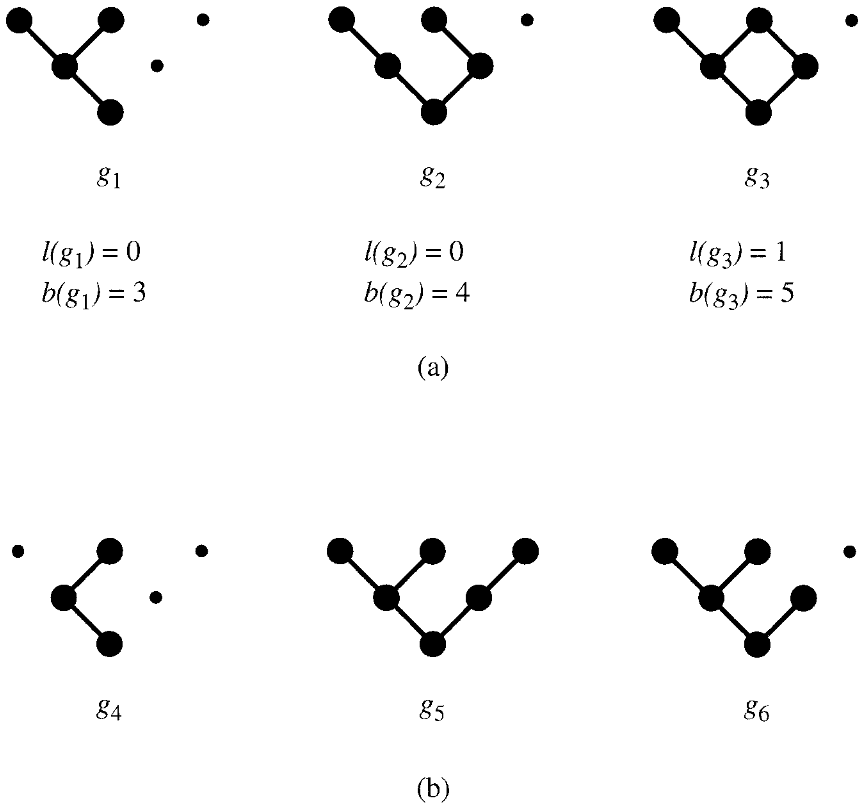

Fig. 3. (a) Three distinct graphs, g_1, g_2 and g_3, in $\mathcal{G}_2(-2,0)$. The quantities $\ell(g_i)$, $b(g_i)$ denote the number of loops and bonds in $g_i \in \mathcal{G}_2(-2,0)$. (b) The graphs, g_4, g_5 and g_6, are *not* included in $\mathcal{G}_2(-2,0)$ by the following reasons. g_4: the site $(x,n)=(-2,2)$ is not connected to the origin O. g_5: the site $(2,2)$ is connected to O. g_6: a path from O to $(1,1)$ is terminated at $(1,1)$ and does not reach any site at the level $n=2$.

If $p=0$, then $P_n(x_1, x_2,..., x_k)=0$ for any $n \geqslant 1$ and any $(x_i, n) \in V_n$ $(1 \leqslant i \leqslant k)$ with $x_1 < x_2 < \cdots < x_k$, so this case is trivial. Therefore, from now on, we assume $p>0$. The following is our main theorem.

Theorem 1. In the Domany–Kinzel model with $p \in (0,1]$ and $q \in [0,1]$, we have

$$P_n(x_1, x_2,..., x_k) = \sum_{g \in \mathcal{G}_n(x_1, x_2,..., x_k)} \left(\frac{q-2p}{p^2}\right)^{\ell(g)} p^{b(g)} \tag{1.2}$$

where $n \in \{1, 2,...\}$ and $(x_i, n) \in V_n (1 \leqslant i \leqslant k)$ with $x_1 < x_2 < \cdots < x_k$.

As shown in Fig. 3, $\mathcal{G}_2(-2,0)$ has three elements, g_1, g_2, and g_3, and we see $\ell(g_1)=\ell(g_2)=0$, $\ell(g_3)=1$ and $b(g_1)=3$, $b(g_2)=4$, $b(g_3)=5$. Then the right-hand side (RHS) of (1.2) gives

$$\sum_{g\in\mathcal{G}_2(-2,0)}\left(\frac{q-2p}{p^2}\right)^{\ell(g)} p^{b(g)}=p^3+p^4+\left(\frac{q-2p}{p^2}\right)p^5=(1+q)\,p^3-p^4$$

Compared with the direct calculation given above Theorem 1, it is concluded that

$$P_2(-2,0)=\sum_{g\in\mathcal{G}_2(-2,0)}\left(\frac{q-2p}{p^2}\right)^{\ell(g)} p^{b(g)}$$

i.e., Theorem 1 holds in this example.

Here we give some expressions of (1.2) in special cases.

(i) oriented bond percolation ($q=p(2-p)$):

$$P_n(x_1,x_2,\ldots,x_k)=\sum_{g\in\mathcal{G}_n(x_1,x_2,\ldots,x_k)}(-1)^{\ell(g)}\,p^{b(g)} \tag{1.3}$$

(ii) oriented site percolation ($q=p$):

$$P_n(x_1,x_2,\ldots,x_k)=\sum_{g\in\mathcal{G}_n(x_1,x_2,\ldots,x_k)}(-1)^{\ell(g)}\,p^{s(g)-1} \tag{1.4}$$

where $s(g)$ is the number of sites in g. In (1.4), we used Euler's law: $s(g)=b(g)-\ell(g)+1$.

(iii) oriented site-bond percolation ($p=\alpha\beta$ and $q=\alpha(2\beta-\beta^2)$):

$$\begin{aligned}P_n(x_1,x_2,\ldots,x_k)&=\sum_{g\in\mathcal{G}_n(x_1,x_2,\ldots,x_k)}\left(-\frac{1}{\alpha}\right)^{\ell(g)}(\alpha\beta)^{b(g)}\\&=\sum_{g\in\mathcal{G}_n(x_1,x_2,\ldots,x_k)}(-1)^{\ell(g)}\,\alpha^{s(g)-1}\beta^{b(g)}\end{aligned} \tag{1.5}$$

where we also used Euler's law. It should be remarked that in the context of percolation problems the k-point correlation functions defined by (1.1) can be regarded as the point-to-set connectivities, where the set includes k distinct points. The pair connectedness function is a special case with $k=1$ (i.e., a point-to-point connectivity).

Remark. Arrowsmith and Essam studied an *inhomogeneous* site-bond percolation model on an oriented graph G.[3] Let V and A are respectively the site and bond sets of G and consider the case that each site $w\in V$

(resp. each bond $a \in A$) is open with probability p_w (resp. p_a) and is closed with probability $1-p_w$ (resp. $1-p_a$) independently. They studied the coefficients $\{\vec{d}\}$, which are called *directed d-weights*, in the expansion of the pair connectedness function P_{uv} between two sites $u, v \in G$ in the form

$$P_{uv}(G) = \sum_{A' \subseteq A} \vec{d}(G') \prod_{a \in A'} p_a \prod_{w \in V'} p_w \tag{1.6}$$

where G' is the bond set A' with its set of incident sites V'. (See Eq. (2.2) and remarks for oriented graphs below it in ref. 3.) In their paper,[3] they first claimed that $\vec{d}(G')=0$ if G' is not coverable by paths from u to v and then proved that, if $\vec{d}(G') \neq 0$, then

$$\vec{d}(G') = (-1)^{t_{uv}(G')+1} \tag{1.7}$$

with

$$t_{uv}(G') = |A'| - |V'| + 2 \tag{1.8}$$

(See Theorem 4 and Lemma 5 in ref. 3.) If we consider a homogeneous site-bond percolation model on an oriented square lattice $\mathbf{T}$ with $u=(0,0)$ and $v=(x_1, n)$, in which $p_w=\alpha$ for any site $w \neq u$ and $p_u=1$, and $p_a=\beta$ for any bond a in $\mathbf{T}$, then we can identify the results (1.6)–(1.8) by Arrowsmith and Essam with the special case with $k=1$ of our formula (1.5), since in this case $\vec{d}(G') \neq 0$ iff $G' \in \mathcal{G}_n(x_1)$ by our definition of $\mathcal{G}_n(x_1)$ and the fact that $t_{uv}(G')+1 = b(G') - s(G') + 3 = b(G') - s(G') + 1 (\mathrm{mod}\ 2) = \ell(G') (\mathrm{mod}\ 2)$, where we have used Euler's law at the last equality.

In the next section, we will give a proof of Theorem 1. Our proof is given by induction with respect to the level n. Moreover, in a case (see case (ii) in Section 2), we will perform induction on the size of a cluster k at a fixed level.

The formulae (1.3)–(1.5) can be called *low-density expansion formulae*, since the RHS's are power series with respect to the concentration of site and/or that of bond. Such low-density expansion formulae have been used for calculating series-expansions for connectedness functions for percolation models.[4, 5] It should be noted that recently Jensen introduced a new algorithm to calculate long series for oriented percolation models, in which he used the Arrowsmith–Essam formula.[6, 7] He reported that the growth of computational complexity of this new algorithm is exponential but its growth factor is much smaller than that of the previous best algorithm and the series for the bond (site) percolation on the oriented square lattice was extended to order 171 (158). Such a long series has not yet been calculated for general cases of the Domany–Kinzel model. One of the applications of

Theorem 1 is to use it as a graphical expansion formula to calculate correlation functions $P_n(x_1, x_2,..., x_k)$ as a power series with respect to p and $\gamma = (q-2p)/p^2$ for the Domany–Kinzel model. Another application of Theorem 1 is to use it to prove a theorem (Theorem 5), which shows that the correlation functions of the Domany–Kinzel model are obtained as an $m \to 0$ limit of the generating functions of the system of m interacting random walkers called *m friendly walkers* (FW).[8, 9] Definitions of the system of FW with two parameters and the relation of it to the Domany–Kinzel model will be given in Section 3. Some comments on the present results are given in Section 4.

2. PROOF OF THEOREM 1

This section is devoted to a proof of Theorem 1. The basic idea is to use the induction with respect to the level n. First the collection of all sets included in V_n, which is denoted by $\mathscr{U}_n$, is divided into the following three disjoint collections: $\mathscr{U}_n = \mathscr{U}_n^{(1)} \cup \mathscr{U}_n^{(2)} \cup \mathscr{U}_n^{(3)}$:

(i) $\mathscr{U}_n^{(1)} = \{\{(x_1, n), (x_2, n),..., (x_k, n)\} \in \mathscr{U}_n : x_{i+1} - x_i \geqslant 4 \quad (i = 1,..., k-1)\}$,

(ii) $\mathscr{U}_n^{(2)} = \{\{(x_1, n), (x_2, n),..., (x_k, n)\} \in \mathscr{U}_n : x_{i+1} - x_i = 2 \quad (i = 1,..., k-1)\}$,

(iii) $\mathscr{U}_n^{(3)} = \mathscr{U}_n \backslash (\mathscr{U}_n^{(1)} \cup \mathscr{U}_n^{(2)})$.

First we prove Theorem 1 in case (i), $\{(x_1, n),..., (x_k, n)\} \in \mathscr{U}_n^{(1)}$. Compared with case (ii), case (i) is easier to prove. Next we will consider case (ii). Since the proof of case (iii), $\{(x_1, n),..., (x_k, n)\} \in \mathscr{U}_n^{(3)}$, is the combination of the proofs of cases (i) and (ii), we can omit to show it here.

2.1. Case (i)

We assume that the formula (1.2) holds at a certain level n and then we will prove that it holds also at the level $n+1$. We start by demonstrating a simple and typical example.

Example 1. We assume that $n \geqslant 3$. Let $k = 2$ and take $x_1 = -2$, $x_2 = 2$ at the level $n+1$. As shown in Fig. 4a, we let $y_1 = -3$, $y_2 = -1$, $y_3 = 1$, $y_4 = 3$ on the level n. We begin with the left-hand side (LHS) of (1.2).

$$
\begin{aligned}
&\text{LHS of (1.2) with a level } n+1\\
&\quad = P_{n+1}(x_1, x_2)\\
&\quad = q^2 P_n(y_1, y_2, y_3, y_4)\\
&\qquad + qp\{P_n(y_1, y_2, y_3) - P_n(y_1, y_2, y_3, y_4)\}\\
&\qquad + qp\{P_n(y_1, y_2, y_4) - P_n(y_1, y_2, y_3, y_4)\}\\
&\qquad + pq\{P_n(y_1, y_3, y_4) - P_n(y_1, y_2, y_3, y_4)\}\\
&\qquad + pq\{P_n(y_2, y_3, y_4) - P_n(y_1, y_2, y_3, y_4)\}\\
&\qquad + p^2\{P_n(y_1, y_3) - P_n(y_1, y_2, y_3)\\
&\qquad - P_n(y_1, y_3, y_4) + P_n(y_1, y_2, y_3, y_4)\}\\
&\qquad + p^2\{P_n(y_1, y_4) - P_n(y_1, y_2, y_4)\\
&\qquad - P_n(y_1, y_3, y_4) + P_n(y_1, y_2, y_3, y_4)\}\\
&\qquad + p^2\{P_n(y_2, y_3) - P_n(y_1, y_2, y_3)\\
&\qquad - P_n(y_2, y_3, y_4) + P_n(y_1, y_2, y_3, y_4)\}\\
&\qquad + p^2\{P_n(y_2, y_4) - P_n(y_1, y_2, y_4)\\
&\qquad - P_n(y_2, y_3, y_4) + P_n(y_1, y_2, y_3, y_4)\}\\
&\quad = (q-2p)^2\, P_n(y_1, y_2, y_3, y_4)\\
&\qquad + p(q-2p)\{P_n(y_1, y_2, y_3) + P_n(y_1, y_2, y_4)\\
&\qquad + P_n(y_1, y_3, y_4) + P_n(y_2, y_3, y_4)\}\\
&\qquad + p^2\{P_n(y_1, y_3) + P_n(y_1, y_4) + P_n(y_2, y_3) + P_n(y_2, y_4)\}
\end{aligned}
$$

In the second equality, we have used an inclusion-exclusion argument: For example,

$$
\begin{aligned}
P(\eta(y_1, n) = \eta(y_3, n) = 1, \eta(y_2, n) = \eta(y_4, n) = 0)&\\
= P_n(y_1, y_3) - P_n(y_1, y_2, y_3) - P_n(y_1, y_3, y_4)&\\
+ P_n(y_1, y_2, y_3, y_4)&
\end{aligned}
$$

Next we consider the RHS of (1.2). Let $\gamma = (q-2p)/p^2$. By definition of the set of graphs $\mathscr{G}_n(x_1, x_2,..., x_k)$,

$$
\sum_{g \in \mathscr{G}_{n+1}(x_1, x_2)} \gamma^{\ell(g)} p^{b(g)} = \sum_{A \subset \{y_1, y_2, y_3, y_4\},\, A \neq \varnothing} c(A) \sum_{g' \in \mathscr{G}_n(A)} \gamma^{\ell(g')} p^{b(g')}
$$

where the coefficients $c(A)$ are the functions of γ and p. For example, let $A = \{y_1, y_2, y_3\}$. As shown in Fig. 4b, g' is a graph connecting the origin and $\{(y_1, n), (y_2, n), (y_3, n)\}$. In order to make a graph $g \in \mathscr{G}_{n+1}(x_1, x_2)$ by adding bonds to g', we have to link (y_1, n) to $(x_1, n+1)$, (y_2, n) to $(x_1, n+1)$, and (y_3, n) to $(x_2, n+1)$ by bonds as shown in Fig. 4c. (If we do not link (y_2, n) to $(x_1, n+1)$, for example, then the path from the origin to (y_2, n) is terminated there, i.e., the site (y_2, n) is *dangling*, and the obtained graph is not included in $\mathscr{G}_{n+1}(x_1, x_2)$.) These three new bonds give a factor p^3 and a vertex $\overline{y_1 x_1 y_2}$ indicates creation of a new loop in g, which gives a factor γ^1. Then we have $c(y_1, y_2, y_3) = \gamma^1 p^3$. Following the same kind of consideration for each $A \subset \{y_1, y_2, y_3, y_4\}$, $A \neq \emptyset$, we have

$$
\begin{aligned}
&\text{RHS of (1.2) with a level } n+1\\
&= \gamma^2 p^4 \sum_{g \in \mathscr{G}_n(y_1, y_2, y_3, y_4)} \gamma^{\ell(g)} p^{b(g)}\\
&\quad + \gamma^1 p^3 \Bigg\{ \sum_{g \in \mathscr{G}_n(y_1, y_2, y_3)} \gamma^{\ell(g)} p^{b(g)} + \sum_{g \in \mathscr{G}_n(y_1, y_2, y_4)} \gamma^{\ell(g)} p^{b(g)}\\
&\quad + \sum_{g \in \mathscr{G}_n(y_1, y_3, y_4)} \gamma^{\ell(g)} p^{b(g)} + \sum_{g \in \mathscr{G}_n(y_2, y_3, y_4)} \gamma^{\ell(g)} p^{b(g)} \Bigg\}\\
&\quad + \gamma^0 p^2 \Bigg\{ \sum_{g \in \mathscr{G}_n(y_1, y_3)} \gamma^{\ell(g)} p^{b(g)} + \sum_{g \in \mathscr{G}_n(y_1, y_4)} \gamma^{\ell(g)} p^{b(g)}\\
&\quad + \sum_{g \in \mathscr{G}_n(y_2, y_3)} \gamma^{\ell(g)} p^{b(g)} + \sum_{g \in \mathscr{G}_n(y_2, y_4)} \gamma^{\ell(g)} p^{b(g)} \Bigg\}\\
&= (q-2p)^2\, P_n(y_1, y_2, y_3, y_4)\\
&\quad + p(q-2p)\{P_n(y_1, y_2, y_3) + P_n(y_1, y_2, y_4)\\
&\quad + P_n(y_1, y_3, y_4) + P_n(y_2, y_3, y_4)\}\\
&\quad + p^2\{P_n(y_1, y_3) + P_n(y_1, y_4) + P_n(y_2, y_3) + P_n(y_2, y_4)\}
\end{aligned}
$$

The second equality comes from the assumption of induction. We can find that any coefficients of $P_n(A)$ with $A \subset \{y_1, y_2, y_3, y_4\}$ and $A \neq \emptyset$ in the LHS are equal to the corresponding coefficients in the RHS and we can say that Theorem 1 holds in Example 1.

For any $\{(x_1, n), (x_2, n),..., (x_k, n)\} \in \mathscr{U}_n^{(1)}$, define $z_i = x_i - 1$, $z_i^* = x_i + 1$. In the above example, $x_1 = -2$, $x_2 = 2$, $z_1 = -3 = y_1$, $z_1^* = -1 = y_2$, $z_2 = 1 = y_3$, $z_2^* = 3 = y_4$. By the definitions of z_i and z_i^*, we see that $z_i^* - z_i = 2$ for any $i \in \{1, 2,..., k\}$. If we assume the formula (1.2) at a level n, then

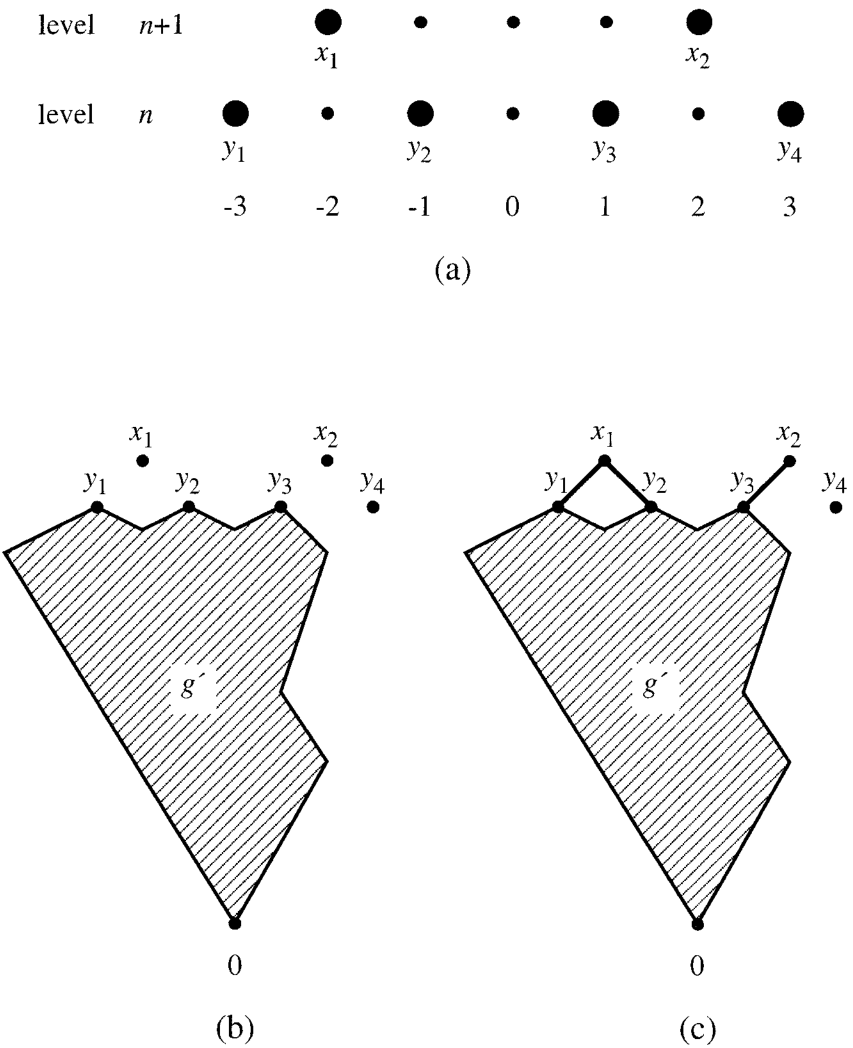

Fig. 4. Figures for Example 1. In (b) and (c), g' denote a graph in $\mathscr{G}_n(y_1, y_2, y_3)$. Remark that all sites (y_i, n), $i = 1, 2, 3$, are connected to the origin O, but the site (y_4, n) is not connected to O in g' by definition of $\mathscr{G}_n(y_1, y_2, y_3)$.

both sides of this formula at a level $n+1$ can be expanded and expressed by linear combinations of $P_n(A)$ with $A \subset \{z_1, z_1^*,..., z_k, z_k^*\}$, $A \neq \varnothing$, as we have found in Example 1. Then what we have to do is to check that any coefficients of $P_n(A)$ in the expansion of the LHS of (1.2) at a level $n+1$ is equal to the corresponding coefficient in the expansion of the RHS of it.

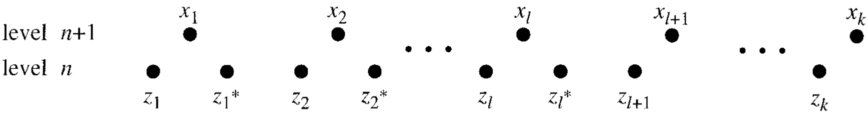

Fig. 5. A figure for $P_n(z_1, z_1^*,..., z_l, z_l^*, z_{l+1},..., z_k)$, where $z_i = x_i - 1$ for $1 \leqslant i \leqslant k$ and $z_i^* = x_i + 1$ for $1 \leqslant i \leqslant l$. It is straightforward to prove the equality for all other coefficients by the same kind of argument. Therefore the proof in the case (i) is completed.

Here we show how to prove the equality using the coefficient of the following type of $P_n(A)$ (see Fig. 5):

$$P_n(z_1, z_1^*, z_2, z_2^*,..., z_l, z_l^*, z_{l+1}, z_{l+2},..., z_k)$$

In the LHS of (1.2) with a level $n+1$, we can calculate the coefficient of this type of $P_n(A)$ as

$$\sum_{m=0}^{l} \binom{l}{m} (-1)^m\, 2^m p^{m+k-l} q^{l-m} = p^{k-l}(q-2p)^l \tag{2.1}$$

On the other hand, in order to make a graph in $\mathscr{G}_{n+1}(x_1, x_2,..., x_k)$ from a graph $g' \in \mathscr{G}_n(z_1, z_1^*,..., z_l, z_l^*, z_{l+1}, z_{l+2},..., z_k)$, we have to link (z_i, n) to $(x_i, n+1)$, $1 \leqslant \forall i \leqslant k$, and (z_i^*, n) to $(x_i, n+1)$, $1 \leqslant \forall i \leqslant l$, by bonds. By this addition of bonds, the number of loops should increase by l and the number of bonds by $2l + (k-l)$. Then we have

$$\begin{aligned} &\text{coefficient of } P_n(z_1, z_1^*,..., z_l, z_l^*, z_{l+1}, z_{l+2},..., z_k) \\ &\quad \text{in the RHS of (1.2) at a level } n+1 \\ &\quad = \left(\frac{q-2p}{p^2}\right)^l p^{2l+(k-l)} \\ &\quad = (q-2p)^l\, p^{k-l} \end{aligned} \tag{2.2}$$

Then the equality of the coefficients of $P_n(z_1, z_1^*,..., z_l, z_l^*, z_{l+1}, z_{l+2},..., z_k)$ in the expansions at the level $n+1$ is concluded if we assume (1.2) at a level n.

2.2. Case (ii)

Before proving Theorem 1 in the case (ii) in general, we give a next simple example for better understanding of our proof.

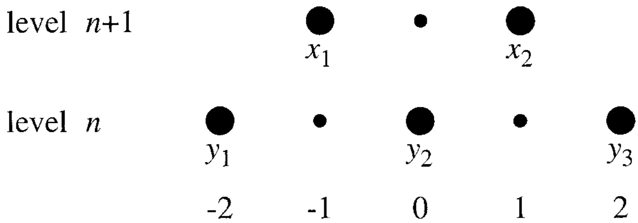

Fig. 6. A figure for Example 2.

Example 2. We assume that $n \geqslant 2$. Let $k=2$, $x_1=-1$, $x_2=1$ at a level $n+1$ and $y_1=-2$, $y_2=0$, $y_3=2$ at a level n as shown in Fig. 6. First we consider the LHS of (1.2).

LHS of (1.2) with a level $n+1$

$$\begin{aligned}
&= P_{n+1}(x_1, x_2)\\
&= q^2 P_n(y_1, y_2, y_3) + pq\{P_n(y_1, y_2) - P_n(y_1, y_2, y_3)\}\\
&\quad + pq\{P_n(y_2, y_3) - P_n(y_1, y_2, y_3)\} + p^2\{P_n(y_1, y_3) - P_n(y_1, y_2, y_3)\}\\
&\quad + p^2\{P_n(y_2) - P_n(y_1, y_2) - P_n(y_2, y_3) + P_n(y_1, y_2, y_3)\}\\
&= q(q-2p)\, P_n(y_1, y_2, y_3) + p(q-p)\{P_n(y_1, y_2) + P_n(y_2, y_3)\}\\
&\quad + p^2\{P_n(y_1, y_3) + P_n(y_2)\}
\end{aligned}$$

where we have used again an inclusion-exclusion argument in the second equality. Next we consider the RHS of (1.2). Let $\gamma = (q-2p)/p^2$.

RHS of (1.2) with a level $n+1$

$$\begin{aligned}
&= (\gamma^2 p^4 + \gamma^1 p^3 + \gamma^1 p^3) \sum_{g \in \mathscr{G}_n(y_1, y_2, y_3)} \gamma^{\ell(g)} p^{b(g)}\\
&\quad + (\gamma^1 p^3 + \gamma^0 p^2) \sum_{g \in \mathscr{G}_n(y_1, y_2)} \gamma^{\ell(g)} p^{b(g)}\\
&\quad + (\gamma^1 p^3 + \gamma^0 p^2) \sum_{g \in \mathscr{G}_n(y_2, y_3)} \gamma^{\ell(g)} p^{b(g)}\\
&\quad + \gamma^0 p^2 \sum_{g \in \mathscr{G}_n(y_1, y_3)} \gamma^{\ell(g)} p^{b(g)} + \gamma^0 p^2 \sum_{g \in \mathscr{G}_n(y_2)} \gamma^{\ell(g)} p^{b(g)}\\
&= (\gamma^2 p^4 + \gamma^1 p^3 + \gamma^1 p^3)\, P_n(y_1, y_2, y_3) + (\gamma^1 p^3 + \gamma^0 p^2)\, P_n(y_1, y_2)\\
&\quad + (\gamma^1 p^3 + \gamma^0 p^2)\, P_n(y_2, y_3) + \gamma^0 p^2 P_n(y_1, y_3) + \gamma^0 p^2 P_n(y_2)\\
&= q(q-2p)\, P_n(y_1, y_2, y_3) + p(q-p)\, P_n(y_1, y_2) + p(q-p)\, P_n(y_2, y_3)\\
&\quad + p^2 P_n(y_1, y_3) + p^2 P_n(y_2)
\end{aligned}$$

The assumption of induction gives the second equality. So we have shown the validity of Theorem 1 also in this example.

In general, for $\{(x_1, n),..., (x_k, n)\} \in \mathcal{U}_n^{(2)}$ both sides of formula (1.2) at a level $n+1$ can be expanded and expressed by linear combinations of $P_n(A)$ with $A \subset \{y_1, y_2,..., y_{k+1}\}$, $A \neq \emptyset$, where $y_1 = x_1 - 1$, $y_2 = x_2 - 1,...,$ $y_k = x_k - 1$, and $y_{k+1} = x_k + 1$, if we assume the formula (1.2) at a level n. We have to check that any coefficients of $P_n(A)$ in the expansion of the LHS of (1.2) at a level $n+1$ is equal to the corresponding coefficient in the expansion of the RHS of it.

Let us first explain in detail how to prove the equality of the coefficients of $P_n(y_1, y_2,..., y_{k+1})$, which corresponds to the coefficient of $P_n(y_1, y_2, y_3)$ in Example 2. Then we give a general proof to the equality of the coefficients of $P_n(A)$ with $A \subset \{y_1, y_2,..., y_{k+1}\}$.

Now we introduce the following subset of 0–1 sequences with length $k \in \{1, 2,...\}$.

$$\begin{aligned} S^{(k)} = \{ s = (s(1), s(2),..., s(k)) \in \{0, 1\}^k : s(i) = s(i+1) = 1, \\ \text{or } s(i) \neq s(i+1) \text{ for any } i \in \{1, 2,..., k-1\}\} \end{aligned} \tag{2.3}$$

For example, $s = (1, 0, 1, 1) \in S^{(4)}$, but $s = (0, 0, 1, 1) \notin S^{(4)}$, since 0, 0-sequence is forbidden. For any $s \in S^{(k)}$, let

$$\sharp(m) = \sharp(m : s) = |\{i \in \{1, 2,..., k\} : s(i) = m\}|$$

$$\sharp(m_1 m_2) = \sharp(m_1 m_2 : s) = |\{i \in \{1, 2,..., k-1\} : s(i) = m_1, s(i+1) = m_2\}|$$

where $m, m_1, m_2 \in \{0, 1\}$. Furthermore we define

$$\begin{aligned}
X_k &= \sum_{s \in S^{(k)}} (-1)^{\sharp(0)} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\
X_k^{\bullet} &= \sum_{s \in S^{(k)} : s(1) = 1} (-1)^{\sharp(0)} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\
X_k^{\circ} &= \sum_{s \in S^{(k)} : s(1) = 0} (-1)^{\sharp(0)} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\
X_k^{\bullet/\bullet} &= \sum_{s \in S^{(k)} : s(1) = s(k) = 1} (-1)^{\sharp(0)} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\
X_k^{\bullet/\circ} &= \sum_{s \in S^{(k)} : s(1) = 1, s(k) = 0} (-1)^{\sharp(0)} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\
X_k^{\circ/\bullet} &= \sum_{s \in S^{(k)} : s(1) = 0, s(k) = 1} (-1)^{\sharp(0)} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\
X_k^{\circ/\circ} &= \sum_{s \in S^{(k)} : s(1) = s(k) = 0} (-1)^{\sharp(0)} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)}
\end{aligned} \tag{2.4}$$

and

$$
\begin{aligned}
W_k &= \sum_{s \in S^{(2k)}} p^{\#(1)} \gamma^{(\#(1)-\#(0))/2} \\
W_k^{\bullet} &= \sum_{s \in S^{(2k)} : s(1)=1} p^{\#(1)} \gamma^{(\#(1)-\#(0))/2} \\
W_k^{\circ} &= \sum_{x \in S^{(2k)} : s(1)=0} p^{\#(1)} \gamma^{(\#(1)-\#(0))/2} \\
W_k^{\bullet/\bullet} &= \sum_{s \in S^{(2k)} : s(1)=s(2k)=1} p^{\#(1)} \gamma^{(\#(1)-\#(0))/2} \\
W_k^{\bullet/\circ} &= \sum_{s \in S^{(2k)} : s(1)=1,\, s(2k)=0} p^{\#(1)} \gamma^{(\#(1)-\#(0))/2}
\end{aligned}
\tag{2.5}
$$

where $\gamma = (q-2p)/p^2$. We should remark that the above definitions give

$$
\begin{gathered}
X_k = X_k^{\bullet} + X_k^{\circ}, \qquad X_k^{\bullet} = X_k^{\bullet/\bullet} + X_k^{\bullet/\circ}, \qquad X_k^{\circ} = X_k^{\circ/\bullet} + X_k^{\circ/\circ} \\
W_k = W_k^{\bullet} + W_k^{\circ}, \qquad W_k^{\bullet} = W_k^{\bullet/\bullet} + W_k^{\bullet/\circ}
\end{gathered}
\tag{2.6}
$$

We will use the following lemma.

Lemma 2.

$$
X_{k+1}^{\bullet} = qX_k^{\bullet} + pX_k^{\circ}, \qquad X_{k+1}^{\circ} = -pX_k^{\bullet} \tag{2.7}
$$

where $X_1^{\bullet} = 1$ and $X_1^{\circ} = -1$, and

$$
\begin{aligned}
W_{k+1}^{\bullet} &= (q-p)\, W_k^{\bullet} + (q-2p)\, W_k^{\circ} \\
W_{k+1}^{\circ} &= pW_k^{\bullet} + pW_k^{\circ}
\end{aligned}
\tag{2.8}
$$

where $W_1^{\bullet} = q-p$ and $W_1^{\circ} = p$.

The proof of Lemma 2 immediately follows the definitions of $X_k^{\bullet}$, X_k°, $W_k^{\bullet}$, and W_k°.

First we state that the coefficient of $P_n(y_1,..., y_{k+1})$ in the expansion of the LHS of (1.2) with a level $n+1$ is equal to X_{k+1}. This fact can be confirmed if we notice the following. Each $s \in S^{(k+1)}$ in X_{k+1} represents a *site* configuration at a level n, in which if $s(i)=1$, $\eta(y_i, n)=1$, and if $s(i)=0$, $\eta(y_i, n)=0$ for $1 \leqslant i \leqslant k+1$. As shown in Fig. 7a, each $(1, 0)$ or $(0, 1)$ pair gives a factor p and each $(1, 1)$ pair q, since $\eta(x_i, n+1)=1$, $1 \leqslant \forall i \leqslant k$, by definition of $P_n(x_1, x_2,..., x_k)$. By the inclusion–exclusion argument, factor -1 should be multiplied for each y_i with $s(i)=0$. Then

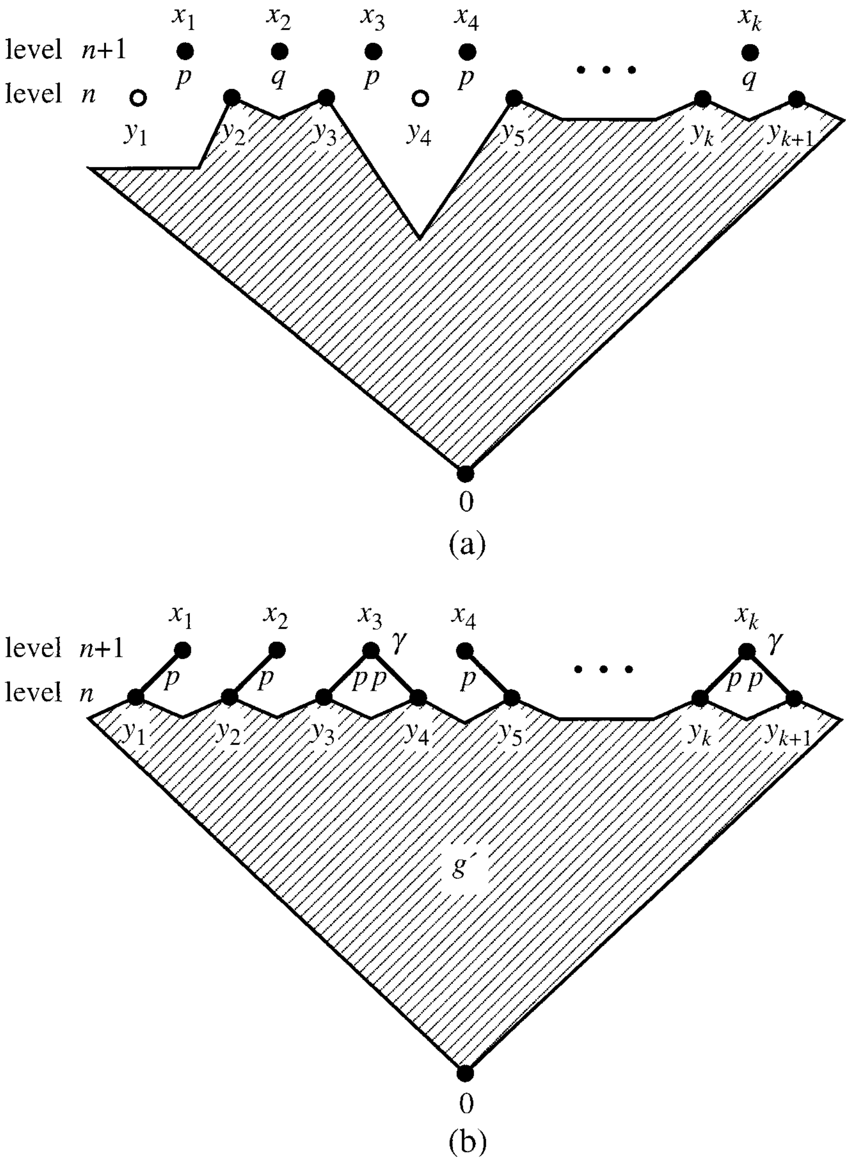

Fig. 7. (a) A site configuration given by a sequence $s = (0, 1, 1, 0, 1,..., 1, 1) \in S^{(k)}$. If $s(i) = 1$ (resp. 0), a full (open) circle is put on the site (y_i, n). If $s(i) = 1$ (resp. $s(i) = 0$), $\eta(y_i, n) = 1$ (resp. 0). (b) A bond configuration given by a sequence $s = (1, 0, 1, 0, 1, 1, 0, 1,..., 1, 1) \in S^{(2k)}$, in which each 1 represents a bond connecting a corresponding pair of sites. Here g' denote a graph in $\mathscr{G}_n(y_1, y_2,..., y_k)$, then remark that all sites (y_i, n), $1 \leqslant i \leqslant k+1$, are connected to the origin in a graph g'.

the contribution from such a site configuration to the coefficient of $P_n(y_1,..., y_{k+1})$ is given by $(-1)^{\#(0)}\, p^{\#(10)+\#(01)} q^{\#(11)}$ and X_{k+1} gives the summation of all those contributions.

Next we state that the coefficient of $P_n(y_1,..., y_{k+1})$ in the expansion of the RHS of (1.2) with a level $n+1$ is equal to $W_k^{\bullet/\bullet}$. In order to confirm this, we first recognize that each $s \in S^{(2k)}$ in $W_k^{\bullet/\bullet}$ represents now a *bond* configuration between the levels n and $n+1$. As shown in Fig. 7b, the leftmost and the rightmost pairs of sites are linked by bonds (corresponding to the condition $s(1) = s(2k) = 1$ in the summation of $W_k^{\bullet/\bullet}$). If $s(2i) = 1$, $1 \leqslant i \leqslant k$, then (y_{i+1}, n) and $(x_i, n+1)$ are connected by a bond, and if $s(2i+1) = 1$, $0 \leqslant i \leqslant k-1$, then (y_{i+1}, n) and $(x_{i+1}, n+1)$ are connected by a bond. Each bond gives a factor p and a vertex $\overline{y_i x_i y_{i+1}}$ gives a factor γ. Then it is easy to see that the contribution from such a bond configuration to the coefficient of $P_n(y_1,..., y_{k+1})$ is given by $p^{\#(1)} \gamma^{(\#(1)-\#(0))/2}$ and $W_k^{\bullet/\bullet}$ gives the summation of all those contributions. Then what we have to do is to prove the identity

$$X_{k+1} = W_k^{\bullet/\bullet} \tag{2.9}$$

for any $k \in \{1, 2,...\}$.

We prove (2.9) by induction with respect to k in the following way. First we consider the case $k = 1$. In this case, we have

$$\begin{aligned} X_2 &= \sum_{s \in \{(0,1),(1,0),(1,1)\}} (-1)^{\#(0)}\, p^{\#(10)+\#(01)} q^{\#(11)} \\ &= (-1)^1\, p^1 q^0 + (-1)^1\, p^1 q^0 + (-1)^0\, p^0 q^1 \\ &= -2p + q \end{aligned}$$

On the other hand,

$$W_1^{\bullet/\bullet} = \sum_{s \in \{(1,1)\}} p^{\#(1)} \gamma^{(\#(1)-\#(0))/2} = p^2 \left(\frac{q-2p}{p^2} \right)^{(2-0)/2} = q - 2p$$

Therefore we have $X_2 = W_1^{\bullet/\bullet}$. That is, (2.9) holds for $k = 1$.

Next we suppose (2.9) to be true for $k = m$;

$$X_{m+1} = W_m^{\bullet/\bullet} \tag{2.10}$$

Under this assumption, we will prove the case $k = m+1$, i.e., $X_{m+2} = W_{m+1}^{\bullet/\bullet}$. By (2.6) and (2.7), we see that

$$\begin{aligned} X_{m+2} &= X^{\bullet}_{m+2} + X^{\circ}_{m+2} \\ &= qX^{\bullet}_{m+1} + pX^{\circ}_{m+1} - pX^{\bullet}_{m+1} \\ &= (q-p)\, X_{m+1} + (2p-q)\, X^{\circ}_{m+1} \end{aligned}$$

On the other hand, the definition of $W^{\bullet/\bullet}_{m+1}$ gives

$$\begin{aligned} W^{\bullet/\bullet}_{m+1} &= p^2 \left(\frac{q-2p}{p^2}\right)^1 W^{\bullet/\bullet}_m + p\left(\frac{q-2p}{p^2}\right)^0 W^{\bullet/\bullet}_m + p^2\left(\frac{q-2p}{p^2}\right)^1 W^{\bullet/\circ}_m \\ &= (q-p)\, W^{\bullet/\bullet}_m + (q-2p)\, W^{\bullet/\circ}_m \end{aligned}$$

Therefore we have

$$X_{m+2} = (q-p)\, X_{m+1} + (2p-q)\, X^{\circ}_{m+1} \tag{2.11}$$

$$W^{\bullet/\bullet}_{m+1} = (q-p)\, W^{\bullet/\bullet}_m + (q-2p)\, W^{\bullet/\circ}_m \tag{2.12}$$

When $q = 2p$, combining (2.10), (2.11) with (2.12) gives the desired conclusion immediately;

$$X_{m+2} = W^{\bullet/\bullet}_{m+1}$$

So, from now on, we assume $q \neq 2p$. By (2.10), (2.11) and (2.12), it suffices to show that

$$X^{\circ}_{m+1} = -W^{\bullet/\circ}_m \tag{2.13}$$

The definition of $W^{\bullet/\circ}_m$ gives

$$W^{\bullet/\circ}_m = pW^{\bullet}_{m-1} \tag{2.14}$$

Therefore, from the second equation of (2.7), (2.14) and $p > 0$, we see that (2.13) can be rewritten as

$$X^{\bullet}_m = W^{\bullet}_{m-1} \tag{2.15}$$

To prove (2.15), we need the following corollary of Lemma 2. Let a and b be roots of $x^2 - qx + p^2 = 0$. Note that $q \neq 2p$ gives $a \neq b$.

Corollary 3.

$$\begin{aligned} X^{\bullet}_k &= \frac{1}{a-b}\{(a^k - b^k) - p(a^{k-1} - b^{k-1})\} \\ X^{\circ}_k &= \frac{p}{a-b}\{-(a^{k-1} - b^{k-1}) + p(a^{k-2} - b^{k-2})\} \end{aligned} \tag{2.16}$$

where $X_1^\bullet = 1$ and $X_1^\circ = -1$, and

$$\begin{aligned} W_k^\bullet &= \frac{1}{a-b}\{(a^{k+1}-b^{k+1}) - p(a^k - b^k)\} \\ W_k^\circ &= \frac{p}{a-b}(a^k - b^k) \end{aligned} \tag{2.17}$$

where $W_1^\bullet = q - p$ and $W_1^\circ = p$.

The proof of Corollary 3 is given by a standard argument as follows. For $X_k^\bullet$ and X_k°, (2.7) gives

$$\begin{aligned} \begin{pmatrix} X_{k+1}^\bullet \\ X_{k+1}^\circ \end{pmatrix} &= \begin{pmatrix} q & p \\ -p & 0 \end{pmatrix} \begin{pmatrix} X_k^\bullet \\ X_k^\circ \end{pmatrix} \\ &= \begin{pmatrix} a & b \\ -p & -p \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & b \\ -p & -p \end{pmatrix}^{-1} \begin{pmatrix} X_k^\bullet \\ X_k^\circ \end{pmatrix} \end{aligned} \tag{2.18}$$

We should remark that a and b are eigenvalues of $\left(\begin{smallmatrix} q & p \\ -p & 0 \end{smallmatrix}\right)$. Then (2.18) gives (2.16). Similarly, for $W_k^\bullet$ and W_k° (2.8) gives

$$\begin{aligned} \begin{pmatrix} W_{k+1}^\bullet \\ W_{k+1}^\circ \end{pmatrix} &= \begin{pmatrix} q-p & q-2p \\ p & p \end{pmatrix} \begin{pmatrix} W_k^\bullet \\ W_k^\circ \end{pmatrix} \\ &= \begin{pmatrix} p-a & p-b \\ -p & -p \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} p-a & p-b \\ -p & -p \end{pmatrix}^{-1} \begin{pmatrix} W_k^\bullet \\ W_k^\circ \end{pmatrix} \end{aligned} \tag{2.19}$$

Note that a and b are also eigenvalues of $\left(\begin{smallmatrix} q-p & q-2p \\ p & p \end{smallmatrix}\right)$. So (2.17) follows (2.19).

The desired result (2.15) immediately follows (2.16) and (2.17) in the above corollary. Then we have proved the equality of the coefficients of $P_n(y_1, y_2, ..., y_{k+1})$.

Now we give proofs of equalities for other coefficients $P_n(A)$, $A \subset \{y_1, y_2, ..., y_{k+1}\}$, $A \neq \emptyset$, $A \neq \{y_1, y_2, ..., y_{k+1}\}$. We start with an example with $k = 7$ and $A = \{y_1, y_2, y_3, y_5, y_6, y_7\}$, that is, $y_4 \notin A$, $y_8 \notin A$, as shown in Fig. 8a. Let L and R be the coefficients of $P_n(A)$ in the LHS and RHS of (1.2) at a level $n+1$. For this example, we have

$$L = \sum_{s \in S^{(8)} : s(4) = s(8) = 0} (-1)^{\#(0)-2} p^{\#(10)+\#(01)} q^{\#(11)}$$

and

$$R = \sum_{s \in S^{(11)} : s(1) = s(5) = s(6) = s(11) = 1} p^{\#(1)} \gamma^{\#_0(A)}$$

where

$$\sharp_0(\Lambda) = \sharp_0(\Lambda : s) \equiv |\{i \in \{1, 2, 4, 5\} : s(2i-1) = s(2i) = 1\}|$$

for $s \in S^{(11)}$. As shown in Fig. 8b, each *site* configuration $s \in S^{(8)}$ with restriction $s(4) = s(8) = 0$ in L can be divided into two subconfigurations $s_1 \in S^{(4)}$ with $s_1(4) = 0$ and $s_2 \in S^{(5)}$ with $s_2(1) = s_2(5) = 0$. That is,

$$\begin{aligned} L = &\sum_{s_1 \in S^{(4)} : s_1(4) = 0} (-1)^{\sharp(0) - 1} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\ &\times \sum_{s_2 \in S^{(5)} : s_2(1) = s_2(5) = 0} (-1)^{\sharp(0) - 2} p^{\sharp(10) + \sharp(01)} q^{\sharp(11)} \\ = &-X_4^{\circ} \times X_5^{\circ/\circ} \end{aligned}$$

On the other hand, Fig. 8c shows that each *bond* configuration $s \in S^{(11)}$ with restriction $s(1) = s(5) = s(6) = s(11) = 1$ in R can be divided into two subconfigurations $s_1 \in S^{(5)}$ with $s_1(1) = s_1(5) = 1$ and $s_2 \in S^{(6)}$ with $s_2(1) = s_2(6) = 1$ and we have

$$\begin{aligned} R = &\sum_{s_1 \in S^{(5)} : s_1(1) = s_1(5) = 1} p^{\sharp(1)} \gamma^{\sharp_1(\Lambda)} \times \sum_{s_2 \in S^{(6)} : s_2(1) = s_2(6) = 1} p^{\sharp(1)} \gamma^{\sharp_2(\Lambda)} \\ = &\, p \sum_{s_1 \in S^{(4)} : s_1(1) = 1} p^{\sharp(1)} \gamma^{(\sharp(1) - \sharp(0))/2} \times p^2 \sum_{s_2 \in S^{(4)}} p^{\sharp(1)} \gamma^{(\sharp(1) - \sharp(0))/2} \\ = &\, p W_2^{\bullet} \times p^2 W_2 \end{aligned}$$

where

$$\sharp_1(\Lambda) = \sharp_1(\Lambda : s_1) \equiv |\{i \in \{1, 2\} : s(2i-1) = s(2i) = 1\}|$$

for $s_1 \in S^{(5)}$ and

$$\sharp_2(\Lambda) = \sharp_2(\Lambda : s_2) \equiv |\{i \in \{1, 2\} : s(2i) = s(2i+1) = 1\}|$$

for $s_2 \in S^{(6)}$. Then if the equalities $-X_4^{\circ} = p W_2^{\bullet}$ and $X_5^{\circ/\circ} = p^2 W_2$ hold, then $L = R$ is concluded for this example.

The above calculation demonstrates the fact that (i) for any $A \subset \{y_1, y_2, ..., y_{k+1}\}$ each site and bond configurations in L and R, respectively, can be divided into their subconfigurations so that L is represented by products of $-X_k^{\circ}$ and $X_k^{\circ/\circ}$ and R by products of $p W_k^{\bullet}$ and $p^2 W_k$ and

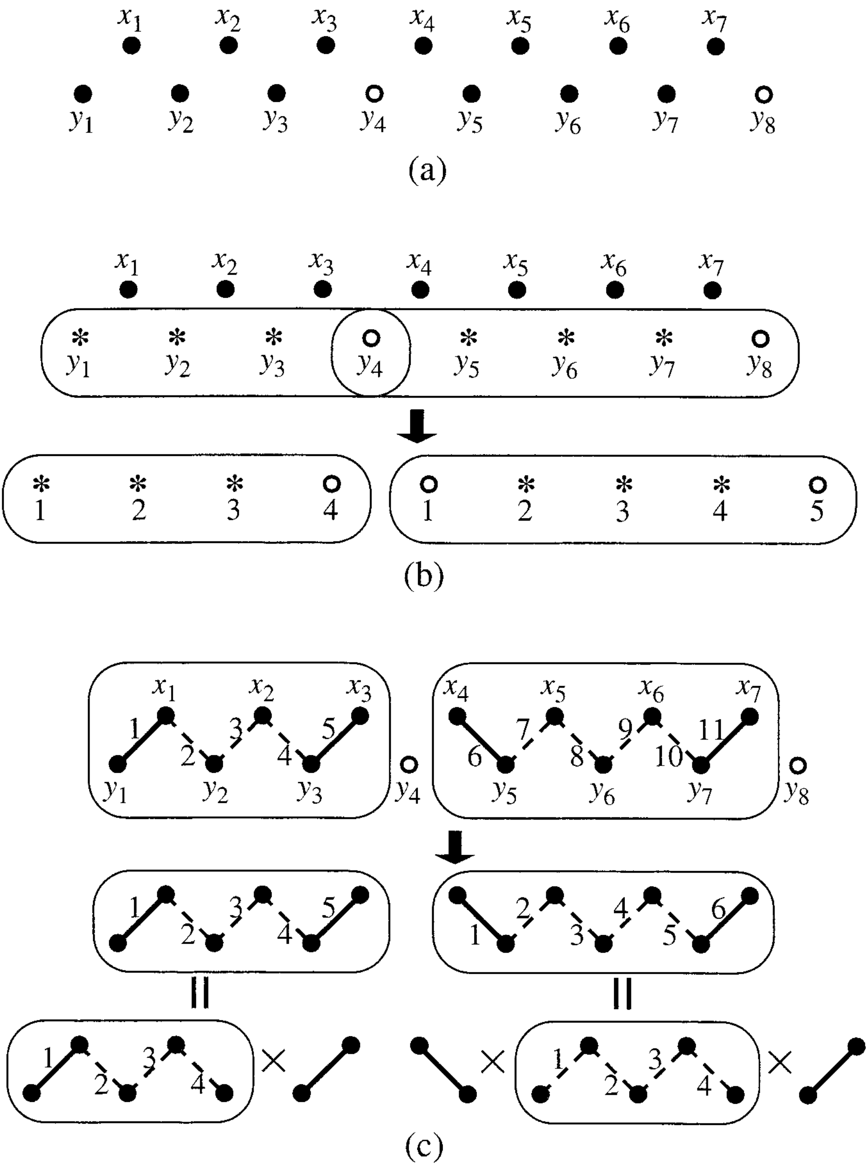

Fig. 8. Figures for the example $A=\{y_1, y_2, y_3, y_5, y_6, y_7\}$. (b) Each site configuration $s \in S^{(8)}$ in L is divided into two subconfigurations. Here sites denoted by open circles should take the value 0 and those denoted by $*$ take 0 or 1. Remark that 0, 0-sequence is forbidden. (c) Each bond configuration $s \in S^{(11)}$ in R is divided into two sub configurations. Here solid lines should be occupied by bonds ($s(i)=1$) and broken lines are empty or occupied by bonds ($s(i)=0$ or 1). Remark that 0, 0-sequence is forbidden.

that (ii) $L=R$ is concluded for any $A \subset \{y_1, y_2,..., y_{k+1}\}$, if the following equalities are satisfied: for any $k \in \{1, 2, 3,...\}$,

$$\text{(a)} \quad -X^{\circ}_{k+2} = pW^{\bullet}_{k}$$

$$\text{(b)} \quad X^{\circ/\circ}_{k+3} = p^2 W_k$$

These equalities are proved as follows.

Proof of (a). This equality immediately follows Corollary 3.

Proof of (b). First remark that we have assumed that $p>0$. Definition (2.4) implies $X^{\circ/\circ}_{k+2} = p^2 X^{\bullet/\bullet}_{k}$. Then (b) is equivalent to

$$X^{\bullet/\bullet}_{k+1} = W_k \tag{2.20}$$

for any $k \in \{1, 2, 3,...\}$. We prove (2.20) by induction with respect to k. It is easy to see that $X^{\bullet/\bullet}_{2} = q = W_1$. Assume that (2.20) holds for $k=m$. Then we consider the case with $k=m+1$. By definition (2.4), we see that

$$X^{\bullet/\bullet}_{m+2} = qX^{\bullet/\bullet}_{m+1} + pX^{\circ/\bullet}_{m+1} \tag{2.21}$$

and

$$\begin{aligned} W_{m+1} &= W^{\bullet}_{m+1} + W^{\circ}_{m+1} \\ &= (q-p)\, W^{\bullet}_{m} + (q-2p)\, W^{\circ}_{m} + pW^{\bullet}_{m} + pW^{\circ}_{m} \\ &= qW_m - pW^{\circ}_{m} \end{aligned} \tag{2.22}$$

where we have used (2.8) of Lemma 2. Since we have assumed (2.20) for $k=m$, what we have to prove is the equality

$$X^{\circ/\bullet}_{m+1} = -W^{\circ}_{m} \tag{2.23}$$

We have already obtained a solution of W°_{m} as the second equation of (2.17) in Corollary 3. Following the calculation similar to that in the proof of Corollary 3, we have

$$X^{\circ/\bullet}_{m+1} = X^{\bullet/\circ}_{m+1} = -\frac{p}{a-b}\,(a^m - b^m)$$

if $q \neq 2p$, where a and b are different roots of $x^2 - qx + p^2 = 0$. Then we can conclude (2.23). When $q = 2p(>0)$, (2.21) and (2.22) are written as

$$X^{\bullet/\bullet}_{m+2} = pX^{\bullet/\bullet}_{m+1} + pX^{\bullet}_{m+1}, \qquad W_{m+1} = pW_m + pW^{\bullet}_{m}$$

respectively. By the assumption of induction and the equality $X^{\bullet}_{m+1} = W^{\bullet}_{m}$, which was proved by Corollary 3, (2.20) with $k = m+1$ is obtained. Then the proof of Theorem 1 in the case (ii) is completed.

Remark. The point of the proof in the case (ii) is to rewrite L (resp. R) by a suitable product of X_k's (resp. W_k's). This *decoupling property* is general, and from this point of view, the proof in the case (i) can be rewritten as follows. Let (2.1) and (2.2) be L and R, respectively, then $L = p^{k-l} X_2^l$ and $R = p^{k-l} (W_1^{\bullet/\bullet})^l$. We have shown $X_2 = q - 2p = W_1^{\bullet/\bullet}$ below (2.9).

The proofs of cases (i) and (ii) are summarized as follows. We prove the equation (1.2) by induction with respect to the level n. We assume that equality at a level n and prove it for a level $n+1$. In proving the equality, we expand the LHS and RHS of the desired equation and show an equality of coefficient of each term in the LHS and that of the corresponding term in the RHS. We have proved the equalities by (a) first showing that coefficients can be represented by products of the quantities X_k's, W_k's and their appropriate modifications defined by (2.4) and (2.5), then (b) identifying the desired equalities between coefficients with the equalities between X_k's, W_k's and their modifications, and (c) proving the equalities between X_k's, W_k's and their modifications by induction with respect to k.

The *decoupling property* that coefficients are represented by using products of X_k's, W_k's and their modifications is general and reduction from the equalities beteen the coefficients to those between X_k's, W_k's and their modifications can be done for any case. Then we can say that it is straightforward to give a proof of Theorem 1 in the case (iii) following the above mentioned three steps as in the cases (i) and (ii). Then the proof of Theorem 1 is completed.

3. APPLICATION OF THEOREM 1: FRIENDLY WALKERS

A system of m friendly walkers (m FW) is an ensemble of paths of m random walkers, who prefer to walk together than walk alone and satisfy the non-crossing condition.[8, 9] It can be regarded as interacting *vicious walkers*[10–14] and will be used as a model system showing interfacial wetting transitions in a two-dimensional $(m+1)$-phase systems.[15] It was first introduced to describe allowed spin configurations of the $(2m+1)$-state chiral Potts model[16, 17] for which a special boundary condition is imposed so that the $m \to 0$ limit of partition function is equal to the percolation probability of the oriented bond percolation on the square lattice.[8] Since the Domany–Kinzel model has two parameters p and q, here we introduce the FW with two parameters. Theorem 1 is very useful to prove the theorem that the $m \to 0$ limit of the generating functions of this generalized FW gives the correlation functions $P_n(x_1, x_2,..., x_k)$ of the Domany–Kinzel model, if we follow the argument of Cardy and Colaiori.[9]

Let us consider an enumeration problem of weighted paths of $m(\geqslant 1)$ walkers on $\mathbf{Z}$ with discrete time. At time $n=0$, all walkers are at the origin and they move simultaneously in time. At each time step $n \to n+1$, each walker steps either to the right or left nearest-neighbor site with equal probability. Two or more walkers can occupy the same site on $\mathbf{Z}$ and they can walk together. We call a set of walkers which occupy the same site a *group*. (When there is only one walker at a site, the group is a singleton.) We label the m walkers by integers $1, 2,..., m$ and the location of the ith walker at time t is written as $z_i(t)$. We impose the *non-crossing condition*,

$$z_1(t) \leqslant z_2(t) \leqslant \cdots \leqslant z_m(t) \qquad \text{for any} \quad t \geqslant 0$$

We introduce two parameters r and τ with $(r, \tau) \in [0, 1]^2$ and the weights of paths of m walkers are determined as follows.

(i) At each time $1 \leqslant t \leqslant n$, count $s_t =$ the number of groups. Then we multiply a weight r^{s_t}.

(ii) At each time $1 \leqslant t \leqslant n$, count $\ell_t =$ the number of distinct sites at which two different *groups* of walkers join together at that time, in which the two groups were separated in a previous time. Then we multiply a height τ^{ℓ_t}.

When two or more walkers walk together along the same path, s_t remains small, but if they tend to walk separately, then s_t becomes large and the weight r^{s_t} becomes small, since $0 \leqslant r \leqslant 1$. This implies that the walkers prefer to walk together than walk alone. By this reason, those walkers are called friendly walkers.[8] Figure 9a shows an example of paths of three $(m=3)$ FW up to $n=12$. We define the k-point generating function of the m FW as a weighted sum of all allowed paths of m walkers up to time $t=n$ with a condition $\{z_1(n), z_2(n),..., z_m(n)\} = \{x_1, x_2,..., x_k\}$, where $x_1 < x_2 < \cdots < x_k$. That is,

$$Z_n(r, \tau; x_1, x_2,..., x_k; m) = \sum_{\text{all paths}} \prod_{t=1}^{n} r^{s_t} \tau^{\ell_t} \tag{3.1}$$

It should be noted that the set of sites (x, n) on $\mathbf{Z}^2$, on which walkers can pass makes $\mathbf{T}$. A path of each walker is a sequence of successive bonds (see Fig. 9a), and a union of paths gives a graph g (see Fig. 9b). That is,

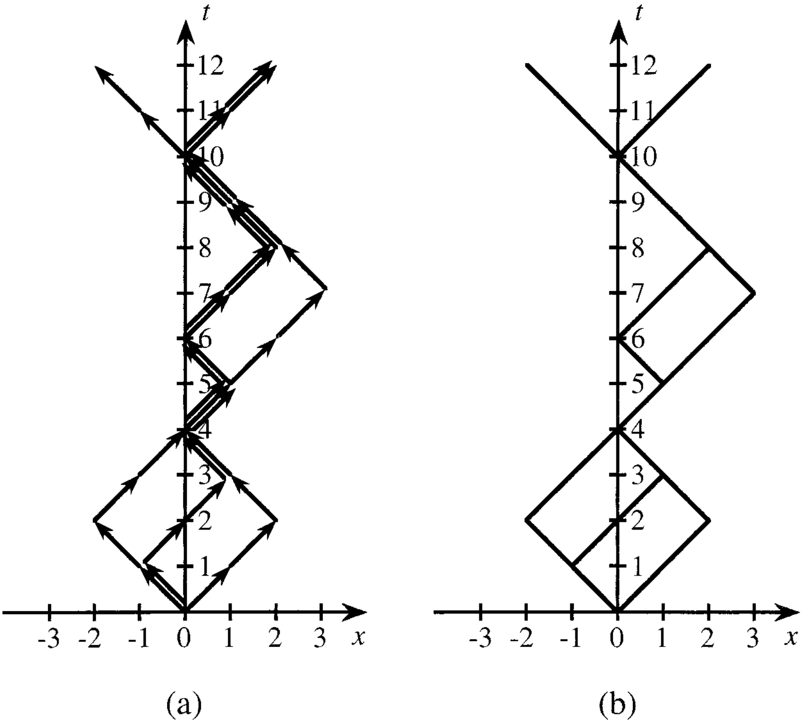

Fig. 9. (a) An example of paths of three ($m=3$) FW up to $n=12$. (b) In this case, a union of paths of three walkers makes a graph $g \in \mathcal{G}_{12}(-2, 2)$. We can see that $s(g)=20$ and $\ell(g)=3$ ($\ell_3=\ell_4=\ell_8=1$ and $\ell_t=0$ for other t).

we consider a system of m walkers such that a union of paths makes a graph $g \in \mathcal{G}_n(x_1,..., x_k)$. By definition, $s(g)=\sum_{t=1}^{n} s_t+1$ and $\ell(g)=\sum_{t=2}^{n} \ell_t$, where s_t and ℓ_t are powers of weights r^{s_t} and τ^{ℓ_t} for the tth step of the FW.

For a given graph $g \in \mathcal{G}_n(x_1,..., x_k)$, however, there may be many different realizations of walking of m FW, which make the same graph g. Let $c(g; m)$ be the number of the distinct paths of the m FW for a given graph $g \in \mathcal{G}_n(x_1, x_2,..., x_k)$. Then (3.1) is rewritten as

$$Z_n(r, \tau; x_1, x_2,..., x_k; m) = \sum_{g \in \mathcal{G}_n(x_1, x_2,..., x_k)} c(g; m)\, r^{s(g)-1} \tau^{\ell(g)} \tag{3.2}$$

So far we have assumed that m is an integer, but we can allow m be any real number, since $c(g; m)$ is a polynomial of m and the generating functions of the m FW depend on m only through $c(g; m)$. Recently Cardy and Colaiori[9] proved the following useful lemma.

Lemma 4 (Cardy and Colaiori[9]). For $g \in \mathcal{G}_n(x_1, x_2,..., x_k)$, define

$$c(g; 0) = \lim_{m \to 0} c(g; m)$$

Then

$$c(g; 0) = (-1)^{k-1+\ell(g)}$$

Combining Theorem 1, (3.2) and Lemma 4 gives the following theorem for the Domany–Kinzel model, if Euler's law is used.

Theorem 5. Define

$$Z_n(r, \tau; x_1, x_2,..., x_k; 0) = \lim_{m \to 0} Z_n(r, \tau; x_1, x_2,..., x_k; m)$$

then

$$P_n(x_1, x_2,..., x_k) = (-1)^{k-1} Z_n\left(p, \frac{2p-q}{p}; x_1, x_2,..., x_k; 0\right)$$

By the definition of $c(g; m)$, if m is fixed as a non-zero finite integer, $c(g; m) = 0$ for the graphs in $\mathcal{G}_n(x_1, x_2,..., x_k)$ which have $m+1$ or more sites at a level V_s, $1 \leqslant s \leqslant n$. This means that $Z_n(r, \tau; x_1, x_2,..., x_k; m)$ given by (3.2) is obtained by a partial sum of the graphs g in $\mathcal{G}_n(x_1, x_2,..., x_k)$.

4. CONCLUDING REMARKS

Now we give some comments on out results. Kinzel studied the three-neighbor model in one-dimension as well as the two-neighbor Domany–Kinzel model,[18] which can be identified with the oriented percolation model on the triangular lattice under some conditions. The same kind of formula as (1.2) may be given also for the triangular lattice, in which the weights of graphs should be determined not only by $b(g)$ and $\ell(g)$ but also by the number of the sites at which three bonds merge into one. We will be able to define the m FW, whose $m \to 0$ limit gives the three-neighbor model. The oriented percolation models and the Domany–Kinzel-type stochastic cellular automata are well-defined in higher dimensions. The generalization of formula (1.2) for the higher dimensional models will be studied.

In the present paper, we have considered only the process ξ_n^0 starting from a singleton at the origin. When we consider processes ξ_n^A starting from other $A \subset \mathbf{Z}$, we can define the following correlation functions in a generalized form,

$$P_n(A, B) = P(\xi_n^A \supset B) \qquad \text{for} \quad A, B \subset \mathbf{Z}$$

In the context of percolation problem, they are identified with set-to-set connectivities.

Generalization of the formula (1.2) for such correlation functions is an interesting future problem.

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